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IMPLEMENTING WITH VETO PLAYERS: A SIMPLE NONCOOPERATIVE GAME

by

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Implementing with veto players: A simple noncooperative game*

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Abstract

The paper adapts a non cooperative game presented by Dagan, Serrano and Volij (1997) for bankruptcy problems to the context of TU veto balanced games. We investigate the relationship between the Nash outcomes of a noncooperative game and solution concepts of cooperative games such as the nucleolus, kernel and the egalitarian core.

1. Introduction

In 1997, Dagan, Serrano and Volij presented a simple noncooperative game for bankruptcy problems. In the game the player with highest claim has a special role. He makes a proposal and the rest of the players in a given order, accept or reject that proposal sequentially. In case of rejection the conflict is solved bilaterally, applying a normative solution concept to a special two-claimant bankruptcy problem. Therefore for any solution defined in the class of two-person bankruptcy problems a noncooperative game can be formed. They prove that, if the solution satisfies certain properties, the outcome of any Nash equilibrium of the game coincides with the *consistent allocation* of the solution used to solve the bilateral conflict. In their conclusions, Dagan, Serrano and Volij (1997) write:

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Thus, constructing consistency based noncooperative models that support consistent cooperative solutions concepts which are not monotonic seems to us a difficult task. Therefore there might be problems in supporting the nucleolus or the Nash bargaining solution on general pies by means of a noncooperative model.

The aim of this paper is to check the validity of this comment. We study a similar noncooperative model in the context of coalitional games with veto players. In our model, a veto player is the proposer and, similarly to Dagan, Serrano and Volij, in case of a negative answer from a responder a bilateral resolution is formulated.

In this bilateral resolution a two-person Davis-Maschler reduced game is defined. We consider two solutions for two-person games.

The first solution applied is the *standard solution* whenever a non negative payoff is provided to the players. Under this resolution of the conflict the nucleolus appears as a candidate to be the Nash outcome of this game. The reason is that in the class of veto balanced games the nucleolus and the prekernel coincide (Arin and Feltkamp, 1997) and the prekernel is the maximal set satisfying the Davis-Maschler reduced game property and *standard solution*¹ (Peleg, 1986).

Given a veto balanced game and its associated noncooperative game we identify the set of allocations to which the outcome of any Nash equilibrium belong. In general, the nucleolus does not belong to this set. On the other hand, any allocation in this set (not necessarily efficient allocations) can be obtained as a result of an equilibrium. Therefore we identify the set of all Nash outcomes. We also find necessary and sufficient conditions for which the nucleolus is a Nash outcome of the game. Mainly, we identify a *monotonicity requirement* that the TU game must meet.

The second solution we consider is the *constrained egalitarian allocation*. This solution is defined by Dutta in 1990 in the class of balanced games. Arin and Inarra (2002) study the concept of constrained egalitarianism in the class of all TU games. With this solution, the elements in the egalitarian core appear as the candidates to be Nash outcomes of the noncooperative game. The reason is that in the class of veto balanced games the egalitarian core is the maximal set satisfying the Davis-Maschler reduced game property and *constrained egalitarian property*. We will see that the set of Nash outcomes and the egalitarian core could

¹By “satisfying standard solution” we mean that the prekernel provides the *standard solution* in any two-person game.

have an empty intersection.

The paper is organized as follows: Section 2 introduces preliminaries on TU games. Section 3 presents the noncooperative model and Section 4 characterizes its Nash outcomes. Section 5 studies the conditions under which the nucleolus is a Nash outcome. The last section analyzes the game when the constrained egalitarian solution is applied.

2. Preliminaries

A *cooperative n -person game in characteristic function form* is a pair (N, v) , where N is a finite set of n elements and $v : 2^N \rightarrow \mathbb{R}$ is a real valued function on the family 2^N of all subsets of N with $v(\emptyset) = 0$. Elements of N are called *players* and the real valued function v the characteristic function of the game. Any subset S of the player set N is called a *coalition*. The number of players in a coalition S is denoted by $|S|$. Given a set of players N and a coalition $S \subset N$ we denote by S^c the set of players of N that are not in S . Generally we shall identify the game (N, v) by its characteristic function v . In this paper we only consider games where the worth of all coalitions is non negative.

A distribution among the players is represented by a real valued vector $x \in \mathbb{R}^N$ where x_i is the payoff assigned by x to player i . A distribution of an amount lower than or equal to $v(N)$ is called a feasible distribution. We denote $\sum_{i \in S} x_i$ by $x(S)$. A distribution satisfying $x(N) = v(N)$ is called an efficient allocation. An efficient allocation satisfying $x_i \geq v(i)$ for all $i \in N$ is called an *imputation* and the set of imputations is denoted by $I(N, v)$. The set of non negative feasible allocations is denoted by $D(N, v)$ and defined as follows

$$D(N, v) = \{x \in \mathbb{R}^N : x(N) \leq v(N) \text{ and } x_i \geq 0 \text{ for all } i \in N\}.$$

The core of a game is the set of imputations that cannot be blocked by any coalition, i.e.

$$C(N, v) = \{x \in I(v) : x(S) \geq v(S) \text{ for all } S \subset N\}.$$

A game with a nonempty core is called a balanced game. A game (N, v) is a *veto-rich game* if it has at least one veto player and the set of imputations is nonempty. A player i is a *veto player* if $v(S) = 0$ for all coalitions where player

i is not present. A balanced game with at least one veto player is called a veto balanced game.

A solution ϕ on a class of games Γ_0 is a correspondence that associates with every game (N, v) in Γ_0 a set $\phi(N, v)$ in \mathbb{R}^N such that $x(N) \leq v(N)$ for all $x \in \phi(N, v)$. This solution is called *efficient* if this inequality holds with equality. The solution is called *single-valued* if the set contains a unique element for every game in the class.

Given a two-person game $(\{1, 2\}, v)$ we use the term *standard solution* for the following vector: $(v(\{1\}) + d, v(\{2\}) + d)$ where $d = \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2}$.

One of the simplest requirements of monotonicity that we ask for in a single-valued solution is aggregate-monotonicity. Let ϕ be a single-valued solution on a class of games Γ_0 . We say that solution ϕ satisfies aggregate-monotonicity property (Meggido, 1974) if the following holds: for all $v, w \in \Gamma_0$, such that for all $S \neq N$, $v(S) = w(S)$ and $v(N) < w(N)$, then for all $i \in N$, $\phi_i(v) \leq \phi_i(w)$.

Given a vector $x \in \mathbb{R}^N$ the *excess of a coalition S with respect to x* in a game (N, v) is defined as $e(S, x) := v(S) - x(S)$. Let $\theta(x)$ be the vector of all excesses at x arranged in non-increasing order of magnitude. The lexicographic order \prec_L between two vectors x and y is defined by $x \prec_L y$ if there exists an index k such that $x_l = y_l$ for all $l < k$ and $x_k < y_k$ and the weak lexicographic order \preceq_L by $x \preceq_L y$ if $x \prec_L y$ or $x = y$.

Schmeidler (1969) introduced the *nucleolus* of a game v , denoted by $\nu(N, v)$, as the unique imputation that lexicographically minimizes the vector of non increasingly ordered excesses over the set of imputations. In formula:

$$\{\nu(N, v)\} = \{x \in I(N, v) \mid \theta(x) \preceq_L \theta(y) \text{ for all } y \in I(N, v)\}.$$

For any game (N, v) with a nonempty imputation set, the nucleolus is a single-valued solution, is contained in the kernel and lies in the core provided that the core is nonempty.

In the class of veto balanced games the kernel, the prekernel and the nucleolus coincide (see Arin and Feltkamp (1997)).

3. The game

Given a veto balanced game (N, v) and an order of players, we define a noncooperative game associated with the TU game and denote it by $G(N, v)$. The game

has n stages and in each stage only one player is playing. In the first stage a veto player is playing and he announces a proposal x^1 that belongs to the set of feasible and non negative allocations of the game (N, v) . In the next stages the responders are playing, each one once at one stage. They have two actions. To accept or to reject. If a player, say i , accepts the proposal x^{t-1} at stage t , he leaves the game with the payoff x_i^{t-1} and for the next stage the proposal x^t coincides with the proposal at $t - 1$, that is x^{t-1} . If player i rejects the proposal then a two-person TU game is formed with the proposer and the player i . In this two-person game the value of the grand coalition is $x_1^{t-1} + x_i^{t-1}$ and the value of the singletons is obtained by applying the Davis-Maschler reduced game² (Davis and Maschler (1965)) given the game (N, v) and the allocation x^{t-1} . The player i will receive as payoff the result of a restricted *standard solution* applied in the two-person game. Once all the responders have played and consequently have received their payoffs the payoff of the veto player is also determined.

Formally, the outcome of playing the game can be described by the following algorithm.

Input : a veto balanced game (N, v) with a veto player, the player 1, and an order in the set of the rest of the players (responders)
Output : a feasible and non negative distribution x .

1. Start with stage 1. The veto player makes a feasible and non negative proposal x^1 (not necessarily an imputation). The superscript denotes at the stage at which the allocation is considered as the proposal in force.
2. In the next stage the first responder says yes or no to the proposal. If he says yes he receives the payoff x_2^1 and $x^2 = x^1$.

²Let (N, v) be a game, $T \subset N$, and consider $T \neq N, \emptyset$ and a feasible allocation x . Then the *Davis-Maschler reduced game* with respect to $N \setminus T$ and x is the game $(N \setminus T, v_x)$ where

$$v_x^{N \setminus T}(S) := \begin{cases} 0 & \text{if } S = \emptyset \\ v(N) - x(T) & \text{if } S = N \setminus T \\ \max_{Q \subset T} \{v(S \cup Q) - x(Q)\} & \text{for all } S \subset N \setminus T. \end{cases}$$

We also denote the game $(N \setminus T, v_x)$ by $v_x^{N \setminus T}$. Note that we define a **modified** Davis-Maschler reduced game where the value of the grand coalition of the reduced game is obtained in a different way. In our case, $v(N \setminus T) = x(N \setminus T)$. If x is efficient the two reduced games coincide.

If he says no he will receive the payoff y_2 where

$$y_2 = \max \left\{ 0, \frac{1}{2}(x_1^1 + x_2^1 - v_{x^1}(\{1\})) \right\} \text{ and}$$

$$v_{x^1}(\{1\}) = \max_{1 \in S \subseteq N \setminus \{2\}} \{v(S) - x^1(S \setminus \{1\})\}$$

$$\text{Now, } x^2 = \begin{cases} x_1^1 + x_2^1 - y_2 & \text{for player 1} \\ y_2 & \text{for player 2} \\ x_i^1 & \text{if } i \neq 1, 2. \end{cases}$$

3. Let the stage t where the k responder plays be given the allocation x^{t-1} . If he says yes he receives the payoff x_k^{t-1} , leaves the game, and $x^t = x^{t-1}$.

If he says no he will receive the payoff

$$y_k = \max \left\{ 0, \frac{1}{2}(x_1^{t-1} + x_k^{t-1} - v_{x^{t-1}}(\{1\})) \right\} \text{ where}$$

$$v_{x^{t-1}}(\{1\}) = \max_{1 \in S \subseteq N \setminus \{t\}} \{v(S) - x^{t-1}(S \setminus \{1\})\}.$$

$$\text{Now, } x^t = \begin{cases} x_1^{t-1} + x_k^{t-1} - y_t & \text{for player 1} \\ y_k & \text{for player } k \\ x_i^{t-1} & \text{if } i \neq 1, k \end{cases}.$$

4. The game ends when stage n is played and we define $x^n(N, v)$ as the vector with coordinates $(x_j^n)_{j \in N}$.

In this game we assume that the conflict between the proposer and a responder is solved bilaterally. In the case of conflict, the players face a two-person TU game that shows the strength of the players given the fact that the rest of the responders do not play. Once the game is formed the allocation proposed for the game is a normative proposal, a kind of restricted *standard solution*³. It is restricted because negative payoffs are not allowed. If the two-person formed game is balanced, the solution will be the *standard solution* that coincides with the prekernel and the nucleolus.

³The main results of the paper do not change if we use the standard solution instead of the restricted standard solution as the concept with which we solve the bilateral conflict. Since our main idea is to discuss simple mechanisms we think it is more credible to assume that no player will accept a negative payoff, a payoff lower than his individual worth.

4. The Nash outcomes

The main question we try to solve is what outcomes we can expect from the equilibria of the game (we call the vector of payoffs associates with a Nash equilibrium a Nash outcome). One might think that the prekernel (that coincides with the nucleolus in the class of veto balanced games) of the game was a good candidate to be a Nash outcome: In case of conflict, in many cases the players solve the situation by applying the prekernel of a game obtained with the Davis-Maschler reduction.

The first example shows that the nucleolus, in general, is not the outcome of equilibrium of the game $G(N, v)$.

Example 4.1. Let $N = \{1, 2, 3, 4, 5, 6\}$ a set of players and consider the following 6-person veto balanced game (N, v) where

$$v(S) = \begin{cases} 1 & \text{if } |S| > 2 \text{ and } 1 \in S \text{ and } S \neq N \\ 3 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Computing the nucleolus⁴ of this game we see that all the players receive the same payoff. It can be immediately checked that if the proposer starts with the following proposal $x^1 = (1, 1, 1, 0, 0, 0)$ after the optimal⁵ answer of the rest by the players the final outcome will be the vector x^1 . Therefore it is clear that in this case the outcome of an equilibrium cannot be the nucleolus.

The main theorem of this section gives the necessary and sufficient conditions to identify all the Nash outcomes of the game. We need some definitions and lemmas before we introduce this main theorem.

Given a game (N, v) and a feasible allocation x we define the complaint of the player i against the player j as follows:

$$f_{ij}(x) = \min_{i \in S \subseteq N \setminus \{j\}} \{x(S) - v(S)\}.$$

⁴Arin and Feltkamp (1997) present an algorithm for computing the nucleolus in the class of TU games with veto players.

⁵If the responders are not playing optimally it is not true that with this proposal the final payoff of the veto player will be at least 1. But it is true if the initial proposal is the vector $(1, 0, 0, 0, 0, 0)$.

The set of *bilaterally balanced allocations* for player i is

$$F_i(N, v) = \{x \in D(N, v) : f_{ji}(x) \geq f_{ij}(x) \text{ for all } j \neq i\}$$

while the *set of optimal allocations* for player i in the set $F_i(N, v)$ is defined as follows:

$$B_i(N, v) = \arg \max_{x \in F_i(N, v)} x_i.$$

Note that since $F_i(N, v)$ is a nonempty (it contains the prekernel⁶) compact set the set $B_i(N, v)$ is nonempty.

Lemma 4.2. *Let (N, v) be a veto balanced TU game and let $G(N, v)$ be its associated noncooperative game. Given a non negative proposal x^{t-1} at stage t the responder playing optimally at this stage, say player i , will reject the proposal x^{t-1} if $f_{i1}(x^{t-1}) < f_{1i}(x^{t-1})$ and will accept the proposal if $f_{i1}(x^{t-1}) > f_{1i}(x^{t-1})$. If $f_{i1}(x^{t-1}) = f_{1i}(x^{t-1})$ the player is indifferent between accepting or rejecting.*

Proof. The responder playing at stage t should compare the amount y_i resulting after rejection with the amount x_i^{t-1} that results after accepting. Note that $x_i^{t-1} = f_{i1}(x^{t-1})$ and $v_{x^{t-1}}(\{1\}) = -f_{1i}(x^{t-1}) + x_1^{t-1}$. Consequently, it holds that $y_i = \max(0, \frac{1}{2}(f_{1i}(x^{t-1}) + f_{i1}(x^{t-1})))$.

Therefore $y_i > x_i^{t-1}$ if and only if $f_{i1}(x^{t-1}) < f_{1i}(x^{t-1})$. ■

Note that if a player i , playing optimally, rejects the proposal x^{t-1} at stage t it holds that $f_{i1}(x^t) = f_{1i}(x^t)$. This is so because

$$\begin{aligned} f_{1i}(x^t) &= f_{1i}(x^{t-1}) - (x_1^{t-1} - x_1^t) \text{ and} \\ x_1^t &= x_1^{t-1} + f_{i1}(x^{t-1}) - y_i. \end{aligned}$$

Combining the two equalities and knowing that $y_i = \max(0, \frac{1}{2}(f_{1i}(x^{t-1}) + f_{i1}(x^{t-1})))$ we get $f_{i1}(x^t) = f_{1i}(x^t)$.

Lemma 4.3. *Let (N, v) be a veto balanced TU game and let $G(N, v)$ be its associated noncooperative game. Given any proposal x^1 , if the responders play best response strategies the final outcome of the game will be an element of $F_1(N, v)$. That is, $x^n \in F_1(N, v)$.*

⁶If we denote the prekernel by PK and (N, v) is a veto balanced games then $PK(N, v) = \bigcap_{i \in N} (F_i(N, v) \cap I(N, v))$. In general, the result is not valid and there exist TU games for which some sets F_i are empty.

Proof. Let i be a responder playing his best response at stage t . If he says yes we have that $f_{i1}(x^{t-1}) \geq f_{1i}(x^{t-1})$ and if he says not we will have that $f_{i1}(x^t) = f_{1i}(x^t)$. It is also clear that if all responders play optimally then $x_1^t \geq x_1^{t+1}$ for all $t \in \{1, \dots, n-1\}$. Note also that in each stage if there is any transfer, is a bilateral transfer from the proposer to a responder. Let l be the responder playing at stage $t+1$. If player l accepts it is clear that $f_{1i}(x^t) = f_{1i}(x^{t+1})$. If player l rejects then either $f_{1i}(x^t) = f_{1i}(x^{t+1})$ or $f_{1i}(x^t) > f_{1i}(x^{t+1})$ depending on which players are contained in the coalition that the proposer is using to complain against the responder i . Therefore $f_{1i}(x^t) \geq f_{1i}(x^{t+1})$ for all $t \in \{1, \dots, n-1\}$ and for all $i \neq 1$. ■

Remark 1. Note that as a consequence of the lemma if the initial proposal belongs to $F_1(N, v)$ then, assuming optimal behavior of the responders, the final proposal will coincide with the initial proposal. That means that the proposer can guarantee a payoff for himself just proposing an allocation of $B_1(N, v)$.

The following theorem is a result of this implication.

Theorem 4.4. Let (N, v) be a veto balanced TU game and let $G(N, v)$ be its associated noncooperative game. Let z be a feasible and non negative allocation. Then z is a Nash outcome if and only if $z \in B_1(N, v)$.

Proof. Let $z \in B_1(N, v)$ and consider the following profile of strategies: z is proposed by the proposer and the responders respond to any proposal by rejecting it if and only if after rejection they increase their payoff. Otherwise they accept. It is immediate that this profile is a Nash equilibrium for which the final payoff vector is z .

Let z be a Nash outcome. By Lemma 4.3 $z \in F_1(N, v)$. Let $k = \max_{x \in F_1(N, v)} x_1$. By definition $z_1 \leq k$. By Remark 1 $z_1 \geq k$. Therefore $z_1 = k$ and consequently $z \in B_1(N, v)$. ■

Remark 2. The result of Theorem 4.4 is independent of the order of the responders.

Analyzing Example 4.1 again we can check that the set $B_1(N, v)$ could contain more than one element. To prove this, first of all we will prove that if z belongs to $B_1(N, v)$ then $z_1 = 1$.

Assume $z = (z_1, z_2, z_3, z_4, z_5, z_6) \in B_1(N, v) \subset F_1(N, v)$ and $z_1 > 1$. Therefore $z_i = f_{i1}(z) \geq f_{1i}(z)$ for all $i \neq 1$. Let i be the non veto player with the lowest payoff according to z . If $z_1 > 1$ it is clear that $f_{1i}(z) > 2z_i \geq z_i$. Therefore it is not true that $z \in F_1(N, v)$.

It can be checked that any vector x , such that $x_1 = 1$ and at least three responders receive 0 will be an element of $B_1(N, v)$.

Note that if the responders are playing optimally, any proposal of the proposer ending in an outcome of $B_1(N, v)$ will be a best strategy for the proposer.

For these reasons we call the elements of the set $B_1(N, v)$ outcomes of equilibrium of the game $G(N, v)$.

We have seen that the elements of $B_1(N, v)$ are not necessarily imputations. In some cases, the set $B_1(N, v)$ does not contain any efficient allocation.

Example 4.5. Let $N = \{1, 2, 3, 4, 5\}$ be a set of players and consider the following 5-person veto balanced games (N, v) and (N, w) where

$$v(S) = \begin{cases} 8 & \text{if } |S| > 3 \\ & \text{and } 1 \in S \\ 0 & \text{otherwise} \end{cases} \quad \text{and } w(S) = \begin{cases} 8 & \text{if } |S| > 3, 1 \in S \\ & \text{and } S \neq N \\ 12 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that for the game $G(N, v)$ the proposal $x^1 = (8, 0, 0, 0, 0)$ is the optimal strategy for the proposer and the final outcome will be x^1 . The result does not depend on the strategies of the responders.

For the game $G(N, w)$ it is still true that the proposal x^1 will end in itself independently of the strategies of the responders⁷. Therefore, any equilibrium should generate an outcome in which the final payoff of the proposer is at least 8. But that is not possible if the proposer is forced to make efficient proposals. The reason is that any imputation z in which z_1 is 8 or higher is not an element of $F_1(N, w)$, as the following argument shows.

Assume $z = (z_1, z_2, z_3, z_4, z_5)$ is an efficient outcome of an equilibrium in the game $G(N, w)$ and that $z_1 \geq 8$. Clearly, $z \neq (12, 0, 0, 0, 0)$. By lemma 4.3 $z \in F_1(N, w)$. Therefore $z_i = f_{i1}(z) \geq f_{1i}(z)$ for all $i \neq 1$. But, if $z_1 \geq 8$ it is true that

⁷This result does not hold if we use the **original** Davis-Maschler reduced two-person games. Since in the game (N, w) the allocation $z = (8, 0, 0, 0, 0)$ is not efficient, $z(S)$ and $w(N) - z(N \setminus S)$ do not coincide.

$f_{1i}(z) \geq \sum_{l \neq 1, i} z_l > z_i$ if i is the responder with lowest payoff. Therefore it is not true that $z \in F_1(N, v)$.

This contradictory example is a direct consequence of the fact that the nucleolus, in the class of veto balanced games, does not satisfy aggregate monotonicity (Arin and Feltkamp, 2004). Section 5 deals with this aspect in detail.

The next example shows that in some cases, the order of the responders influences the outcome of equilibrium. The example does not contradict Remark 2. The set of Nash outcomes of the game $G(N, v)$ is independent of the order of the responders.

Example 4.6. Let $N = \{1, 2, 3, 4\}$ be a set of players and consider the following 4-person veto balanced game (N, v) where

$$v(S) = \begin{cases} 1 & \text{if } |S| > 1, 1 \in S \\ & \text{and } S \neq N \\ 1.5 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Consider the following proposal: $(1.5, 0, 0, 0)$. Given this proposal it can be checked that the final outcome of the game if the players play optimally will be the following: $x_1 = 1$, the player answering first gets 0.25 and the last two responders obtain 0.125 each. Therefore this outcome depends on the order of the responders.

Moreover the final outcome, $(1, 0.25, 0.125, 0.125)$, an element of $B_1(N, v)$, is a Nash outcome. To see this we need to prove that if $z \in B_1(N, v)$ then $z_1 = 1$. Assume in contrast that there is z such that $z \in B_1(N, v)$ and $z = (1 + \varepsilon, z_2, z_3, z_4)$ where $\varepsilon > 0$. Without loss of generality let z_4 be the lowest payoff and z_3 the second lowest payoff.

Since $z \in F_1(N, v)$ the following inequalities hold:

$$f_{31}(z) = z_3 \geq f_{13}(z) = \varepsilon + z_4,$$

$$f_{41}(z) = z_4 \geq f_{14}(z) = \varepsilon + z_3,$$

and combining the two inequalities we have the following contradiction

$$z_4 \geq \varepsilon + z_3 \geq \varepsilon + \varepsilon + z_4.$$

The example shows that the set $B_1(N, v)$ does not satisfy the equal treatment property⁸, that is, equal players do not receive equal payoff. It is clear that the set always contain elements where equal players are treated equally since it is always possible to decrease the payoff of the player with highest payoff until the payoffs are equalized.

The main result of this section (Theorem 4.4) does not change if we consider a non veto player as the proposer. That is, given a TU veto balanced game and its associated noncooperative game $G(N, v)$ where the proposer is player i , not necessarily a veto player, the outcome of any equilibrium of the game will be an element of $B_i(N, v)$. And any element of $B_i(N, v)$ is the result of at least one equilibrium. The proofs should be modified slightly taking into account the following equalities:

$$v_{x^{t-1}}(\{k\}) = -f_{kl}(x^{t-1}) + x_k^{t-1} \text{ and } v_{x^{t-1}}(\{l\}) = -f_{lk}(x^{t-1}) + x_l^{t-1}.$$

And consequently

$$\begin{aligned} y &= \max(0, \frac{1}{2}(x_l^{t-1} + x_k^{t-1} - v_{x^{t-1}}(\{l\}) - v_{x^{t-1}}(\{k\}))) = \\ &= \max(0, \frac{1}{2}(f_{kl}(x^{t-1}) + f_{lk}(x^{t-1}))). \end{aligned}$$

In fact, similar results can be obtained if we take a TU game without veto players as the game with which we formulate the game $G(N, v)$. In this case it is not clear how to determine the proposer. These results depend on the nonemptiness of the sets denoted by $F_i(N, v)$. Non emptiness is guaranteed if $PK(N, v) \subset D(N, v)$. There are games for which those sets are empty.

Example 4.7. Let $N = \{1, 2, 3\}$ be a set of players and consider the following 3-person veto non balanced game (N, v) where

$$v(S) = \begin{cases} 8 & \text{if } |S| > 1, 1 \in S \\ & \text{and } S \neq N \\ 4 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

In this game the sets $F_2(N, v)$ and $F_3(N, v)$ are empty.

⁸A solution ϕ satisfies the **equal treatment property** if for each (N, v) in Γ_0 and for every $x \in \phi(N, v)$ interchangeable players i, j are treated equally, i.e., $x_i = x_j$. Here, i and j are interchangeable if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$.

5. Monotonicity, Nucleolus and Nash outcomes

In this section we prove the following result: The nucleolus (the kernel) of a veto balanced game (N, v) is a Nash outcome of the game $G(N, v)$ if and only if the nucleolus has a *monotonic behavior* with respect to the proposer's (a veto player) payoff.

A game is monotonic with respect to a player's nucleolus if the decreasing of the worth of the grand coalition implies the nonincreasing of the payoff provided by the nucleolus to that player. Formally, referring to veto players

Definition 5.1. A veto balanced game (N, v) is monotonic with respect to the veto players nucleolus if for all (N, w) with a nonempty set of imputations such that for all $S \neq N$, $w(S) = v(S)$ and $w(N) < v(N)$ it holds that $\nu_i(w) \leq \nu_i(v)$ for all $i \in T$, where T is the set of veto players.

Note that the property refers to a fixed game and not to the solution. A game could be monotonic with respect to the veto players nucleolus while the nucleolus is not monotonic in the class of games to which the game belongs.

In the class of veto rich games (games with a veto player and a nonempty set of imputations) the kernel and the nucleolus coincide. Therefore we can define the nucleolus as

$$\nu(N, v) = \{x \in I(N, v); f_{ij}(x) < f_{ji}(x) \implies x_j = 0\}.$$

The next lemma shows that if for a feasible allocation the *bilateral kernel conditions* hold for the veto player and the rest of the players then those *bilateral kernel conditions* hold for any pair of players.

Lemma 5.2. Let (N, v) be a veto balanced TU game where player 1 is a veto player. Let z be a feasible allocation such that $f_{1i}(z) = f_{i1}(z)$ for all $i \neq 1$, $i \in N \setminus T$ and $f_{1i}(z) \leq f_{i1}(z)$ for all $i \neq 1$, $i \in T$ where $T = \{i \in N; z_i = 0\}$. Then: a) $f_{ij}(z) = f_{ji}(z)$ for all $i, j \in N \setminus T$, $i \neq j$. b) $f_{ji}(z) \leq f_{ij}(z)$ for all i, j such that $j \in N \setminus T$.

Proof. Note that $f_{ji}(z) \leq z_j$ and $f_{i1}(z) = z_i$.

a) Assume there exist players i, j such that $f_{1i}(z) = z_i \geq f_{ij}(z) > f_{ji}(z)$. Assume that $f_{ji}(z) = z_j$. Then

$$f_{1i}(z) = z_i \geq f_{ij}(z) > f_{ji}(z) = z_j = f_{1j}(z).$$

If $f_{1i}(z) > f_{1j}(z)$ then the coalition that player 1 is using to complain against player j should contain player i and can therefore be used as a complaint of player i against player j contradicting that $f_{ij}(z) > f_{1j}(z)$.

Therefore $f_{ji}(z) < z_j$. That means that there exists a coalition S such that $j \in S$ and $i \notin S$ for which $z(S) - v(S) = f_{ij}(z)$. But since (N, v) is a veto balanced game such a coalition should contain the veto player 1. And therefore $f_{1i}(z) \leq z(S) - v(S) = f_{ji}(z) < f_{ij}(z) \leq f_{1i}(z)$.

This contradiction ends part a) of the proof.

b) The proof of this part is very similar to the previous one and is therefore omitted. ■

Theorem 5.3. *Let (N, v) be a veto balanced TU game where player 1 is a veto player. Then $\nu(N, v) \in B_1(N, v)$ if and only if the game (N, v) is monotonic with respect to the veto players nucleolus.*

Proof. Assume that the game (N, v) is not monotonic with respect to the veto players nucleolus. That means that there exists a game (N, w) such that for all $S \neq N$, $w(S) = v(S)$ and $w(N) < v(N)$ and it holds that $\nu_1(w) > \nu_1(v)$. Since $\nu(w) \in F_1(v)$ it is clear that $\nu(v) \notin B_1(v)$.

Now assume that $\nu(v) \notin B_1(v)$ and let z be an element of $B_1(v)$. Let $T = \{i \in N; z_i = 0\}$. There are two cases:

a) $f_{1i}(z) = f_{i1}(z)$ for all $i \neq 1$, $i \in N \setminus T$ and $f_{1j}(z) \leq f_{j1}(z)$ for all $j \neq 1$, $j \in T$. By Lemma 5.2 $f_{ij}(z) = f_{ji}(z)$ for all $i, j \in N \setminus T$, $i \neq j$ and $f_{ij}(z) \leq f_{ji}(z)$ for all i, j such that $j \in N \setminus T$ and $i \in N$. Therefore if $z \neq \nu(v)$ it should be because $z(N) < v(N)$. And it is clear that the allocation z is the nucleolus of the game (N, w) where for all $S \neq N$, $w(S) = v(S)$ and $w(N) = z(N)$. We conclude that the game (N, v) is not monotonic with respect to the nucleolus.

b) $f_{1i}(z) < f_{i1}(z)$ for any $i \neq 1$ and $z_i > 0$. Since $f_{i1}(z) = z_i$ by decreasing the payoff of player i we can construct a new allocation y such that $f_{1i}(y) = f_{i1}(y)$ or $f_{1i}(y) < f_{i1}(y)$ and $y_i = 0$. In any case, $z_1 = y_1$ and therefore $y \in B_1(N, v)$.

Now if there exists another player l such that $f_{1l}(y) < f_{l1}(y)$ and $y_l > 0$ we construct a new allocation x such that $f_{1l}(x) = f_{l1}(x)$ or $f_{1l}(x) < f_{l1}(x)$ and $x_l = 0$. Again, $x_1 = y_1$ and $x \in B_1(N, v)$. Repeating this procedure we will end up with an allocation that is the kernel (nucleolus) of a game where the only change with respect to the game (N, v) is the fact that we have decreased the worth of the grand coalition. If q is the final outcome of this procedure, q is the nucleolus of the game (N, v_q) where $v_q(N) = q(N)$ and $v_q(S) = v(S)$ for all $S \neq N$. This is so because of the previous lemma; once the *kernel bilateral conditions* hold between

the veto player and the rest of the players, those *kernel bilateral conditions* hold for any pair of players. Therefore the game (N, v) is not monotonic with respect to the veto players nucleolus. ■

A direct implication of the proof is the following corollary that could be useful for computing $B_1(N, v)$.

Corollary 5.4. *Let (N, v) be a veto balanced TU game where player 1 is a veto player. Then there exist $z \in B_1(N, v)$ and a game (N, w) where for all $S \neq N$, $w(S) = v(S)$ and $w(N) \leq v(N)$ such that $z = \nu(N, w)$.*

The results are not valid for the class of balanced games. The following example is a 4-person balanced game that is monotonic with respect to the nucleolus payoff of player 1. But the nucleolus is not a Nash outcome of the game where player 1 is the proposer.

Example 5.5. *Let $N = \{1, 2, 3, 4\}$ be a set of players and consider the following 4-person balanced game (N, v) where*

$$v(S) = \begin{cases} 2 & \text{if } S \in \left\{ \begin{array}{l} \{1, 2\}, \{1, 3\}, \{1, 4\}, \\ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \end{array} \right\} \\ 1 & \text{if } S \in \{2, 3, 4\} \\ 3 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

Consider player 1 as the proposer. This game is monotonic with respect to player 1's nucleolus. And $\nu(N, v) = (1.8, 0.4, 0.4, 0.4)$. But $z = (2, 1, 0, 0) \in F_1(N, v)$ and therefore $\nu(N, v) \notin B_1(N, v)$.

In the class of balanced games Lemma 5.2 does not apply. The fact that given an allocation the proposer equals the complaint of all the responders does not imply that those complaints are bilaterally equalized among the responders.

This is one of the main characteristics of the mechanism. Only the bilateral complaint between the proposer and the responder matters.

In Serrano (1997) a more complex non-cooperative model is analyzed. In each period a pair of players bargains facing a given status quo (an allocation). Once this bilateral bargaining is solved a new status quo is fixed (with at most two coordinates changed) and a new pair of players is chosen for a bargaining procedure. At the end all pair of players have faced a bargaining procedure.

6. Constrained egalitarianism: Egalitarian core and Nash outcomes

In Section 3 we introduce a noncooperative game where the conflict between the proposer and the responder is solved by applying the standard solution in a special two person game. In this section we modify this noncooperative game slightly: The conflict is solved by applying the constrained egalitarian allocation in a two person game that is constructed as in the model of Section 3. Therefore we omit the formal presentation of the new noncooperative game and we will refer to it as the noncooperative game where two person games are solved by applying *CEA*. The main theorem of this section gives the necessary and sufficient conditions to identify all the Nash outcomes of this new game.

Dutta (1990) defines the constrained egalitarian solution for 2-person balanced games. Let $(\{i, j\}, v)$ be a 2-person balanced game. With no loss of generality, let $v(\{i\}) \leq v(\{j\})$. The *constrained egalitarian solution* of the game, $CE(\{i, j\}, v)$, provides

$$\begin{aligned} CE_j(\{i, j\}, v) & : = \max \left\{ \frac{v(\{i, j\})}{2}, v(\{j\}) \right\}, \\ CE_i(\{i, j\}, v) & = v(\{i, j\}) - CE_j(\{i, j\}, v). \end{aligned}$$

If the two-person game is not balanced we apply the following definition (Arin and Inarra, 2002). Let $(\{i, j\}, v)$ be a 2-person non balanced game. With no loss of generality, let $v(\{i\}) \leq v(\{j\})$ and denote by $d = v(\{i, j\}) - v(\{i\}) - v(\{j\})$. The *constrained egalitarian solution* of the game, $CE(\{i, j\}, v)$, provides

$$\begin{aligned} CE_j(\{i, j\}, v) & : = \max \left\{ \frac{v(\{i, j\})}{2}, v(\{j\}) + d \right\}, \\ CE_i(\{i, j\}, v) & = v(\{i, j\}) - CE_j(\{i, j\}, v). \end{aligned}$$

A solution σ defined on a class of games, Γ_0 , satisfies the constrained egalitarian property, *CEP*, if it coincides with the constrained egalitarian solution for all 2-person games belonging to Γ_0 .

In the class of veto balanced games the egalitarian core (Arin and Inarra⁹, 2001) is the maximal set satisfying the Davis-Maschler reduced game property and

⁹See also Hougard, Peleg and Thorlund-Petersen (2001).

CEP^{10} . The *egalitarian core* of a balanced game (N, v) denoted by $EC(N, v)$, is the set

$$EC(N, v) := \{x \in C(N, v) \mid x_i > x_j \Rightarrow f_{ij}(x) = 0\}.$$

There are different single-valued core solutions that satisfy Davis-Maschler reduced game property and CEP . Clearly, they all belong to the egalitarian core.

The following question arises immediately: Which is the set of Nash outcomes of the noncooperative game where two person games are solved by applying CEA ? Given the characterization of the egalitarian core, the allocations contained in it appear as candidates to be Nash outcomes of the new game. But, in general, this is not the case. We identify the set of Nash outcomes and we provide an example for which this set and the egalitarian core have an empty intersection.

The results and proofs are quite similar to the ones obtained in Section 4. Roughly speaking, the role played by *bilateral kernel* conditions between the proposer and the responders is now played by *egalitarian core bilateral* conditions between the proposer and the responders.

Let (N, v) be a veto balanced game where player 1 is a veto player. The set of *bilaterally egalitarian balanced allocations for player 1* is

$$EF_1(N, v) = \{x \in D(N, v) : x_1 > x_j \Rightarrow f_{1j}(x) \leq 0\}$$

while the *set of optimal allocations for player 1* in the set $EF_1(N, v)$ is defined as follows:

$$EB_1(N, v) = \arg \max_{x \in EF_1(N, v)} x_1.$$

Note that since $EF_1(N, v)$ is a nonempty (it contains the egalitarian core) compact set the set $EB_1(N, v)$ is nonempty.

The following lemma characterizes the optimal behavior of the responders facing a proposal x .

Lemma 6.1. *Let (N, v) be a veto balanced TU game and let $G(N, v)$ be its associated noncooperative game where two person games are solved by applying CEA . Let x^{t-1} be a non negative proposal at stage t and let i be the responder playing optimally at this stage. If $x_i^{t-1} < x_1^{t-1}$ and $f_{1i}(x^{t-1}) > 0$ then player i will reject the proposal x^{t-1} .*

¹⁰The egalitarian set (Arin and Inarra, 2002) is the maximal set satisfying the Davis-Maschler reduced game property and CEP . In the class of veto balanced games the egalitarian set and the egalitarian core coincide.

Proof. The responder playing at stage t should compare the amount y_i that results after rejection with the amount x_i^{t-1} resulting after accepting. Note that $v_{x^{t-1}}(\{1\}) = -f_{1i}(x^{t-1}) + x_1^{t-1}$. If $y_i > x_i^{t-1}$ then

$$y_i = \min \left\{ \frac{x_1^{t-1} + x_i^{t-1}}{2}, x_1^{t-1} + x_i^{t-1} - (f_{1i}(x^{t-1}) + x_1^{t-1}) \right\} = \min \left\{ \frac{x_1^{t-1} + x_i^{t-1}}{2}, x_i^{t-1} + f_{1i}(x^{t-1}) \right\} > x_i^{t-1}.$$

Therefore if $x_i^{t-1} < x_1^{t-1}$ and $f_{1i}(x^{t-1}) > 0$ then $y_i > x_i^{t-1}$.

Note that after rejection, either $x_i^t = x_1^t$ or $x_1^t = -f_{1i}(x^{t-1}) + x_1^{t-1}$ and consequently $f_{1i}(x^t) = -f_{1i}(x^{t-1}) - (x_1^t - (-f_{1i}(x^{t-1}) + x_1^{t-1})) = 0$. ■

With the previous lemma the proof of the following lemma and theorem are almost identical to the proof of Lemma 4.3 and Theorem 4.4.

Lemma 6.2. *Let (N, v) be a veto balanced TU game and let $G(N, v)$ be its associated noncooperative game where two person games are solved applying by CEA. Given any proposal x^1 , and if the responders play best response strategies, the final outcome of the game will be an element of $EF_1(N, v)$. That is, $x^n \in EF_1(N, v)$.*

Theorem 6.3. *Let (N, v) be a veto balanced TU game and let $G(N, v)$ be its associated noncooperative game where two person games are solved applying by CEA. Let z be a feasible and non negative allocation. Then z is a Nash outcome if and only if $z \in EB_1(N, v)$.*

In general, the set of Nash outcomes does not need to contain egalitarian single-valued solutions.

Example 6.4. *Let $N = \{1, 2, 3\}$ be a set of players and consider the following 3-person veto non balanced game (N, v) where*

$$v(S) = \begin{cases} 8 & \text{if } |S| > 1, 1 \in S \\ & \text{and } S \neq N \\ 12 & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

It can immediately be checked that $EB_1(N, v) = \{(8, 0, 0)\}$. On the other hand, the egalitarian core of this game is $EC(N, v) = \{(4, 4, 4)\}$. Therefore the

egalitarian single-valued solutions cannot be obtained as Nash outcomes of the noncooperative game associated with (N, v) . Egalitarian single-valued solutions do not satisfy the aggregate-monotonicity property in the class of veto balanced games.

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